Cite as:
A. Orłowski and L.J. Chmielewski. Ulam Spiral and Prime-Rich Polynomials. In L. Chmielewski et al., editors, *Computer Vision and Graphics. Proc. ICCVG 2018, volume 10114 of Lecture Notes in Computer Science*, pages 522-533, Warsaw, Poland, 17-19 Sep 2018. Springer. doi:10.1007/978-3-030-00692-1_45.

Ulam Spiral and Prime-Rich Polynomials

Arkadiusz Orłowski and Leszek J. Chmielewski

Faculty of Applied Informatics and Mathematics (WZIM),
Warsaw University of Life Sciences (SGGW), Poland
ul. Nowoursynowska 159, 02-775 Warsaw, Poland
{arkadiusz_orlowski,leszek_chmielewski}@sggw.pl
http://www.wzim.sggw.pl

Abstract. The set of prime numbers visualized as Ulam spiral was considered from the image processing perspective. Sequences of primes forming line segments were detected with the special version of the Hough transform. The polynomials which generate the numbers related to these sequences were investigated for their potential richness in prime numbers. One of the polynomials which generates the numbers forming the 11-point sequence was found exceptionally prime-rich, although it was not the longest sequence found. This polynomial is $4n^2 - 1260n + 98827$ and it generates 613 primes (20 of them with the minus sign) for the first 1000 non-negative integers as arguments. This is more than generated by some other well-known prime-rich polynomials, the Euler one included.

Keywords: Ulam spiral · line segment · prime-rich · polynomial · Hough transform

1 Introduction

The set of prime numbers is a subject of intensive research and is still hiding mysteries (cf. [20]). It is interesting from the point of view of many subdomains of mathematics, including the theoretical number theory on one side and the practical cryptography on the other. One of the ways the prime numbers can be visualized is the Ulam spiral, first devised in 1963 [21] and published in 1964 [17]. It is also called prime spiral [19]. The spiral containing the prime numbers and the remaining numbers forming a square, with the number 1 in the center, will be called the Ulam square.

The visual appearance of lines, mostly diagonal, and forming an asymmetric pattern, is a striking feature of the Ulam spiral. This feature became the subject of our studies. We were interested in whether the lines which seem to be visible with a human eye were actually present there, and if so, what were their properties. We wanted to check for the presence of sequences of immediately neighboring points as well as those mutually placed at a longer range, forming sequences of points, or lines, inclined at a specified angle. We wished not to limit the search to lines inclined by such specific angles as 0, 45 or 90 degrees, but to search for lines at arbitrary angles realizable on the square grid. Therefore, we

L. J. Chmielewski et al. (Eds.): ICCVG 2018, LNCS 11114, pp. 522-533, 2018.
https://doi.org/10.1007/978-3-030-00692-1_45

actually went beyond the limit imposed by the noticeability of lines to a human eye. However, we wished to apply image processing techniques, to objectivize the visual effects characteristic to humans.

To detect and describe line segments the Hough transform (HT) can be used. The Ulam square is a strictly defined mathematical object, not a projection of a real-world object containing physical lines. Therefore, a special version of HT should be used. The analysis of digital lines with the HT was described in [7,12], where all the lines, being digitizations of a mathematical straight line representable in an image, were considered. This *digital* HT was compared to te conventional *analog* HT in [11]. Our approach was different, due to that the lines in the Ulam square are not approximations, but are the sequences of points having strictly specified ratios of coordinate increments. For such lines, we have proposed a relatively simple detection method in [3].

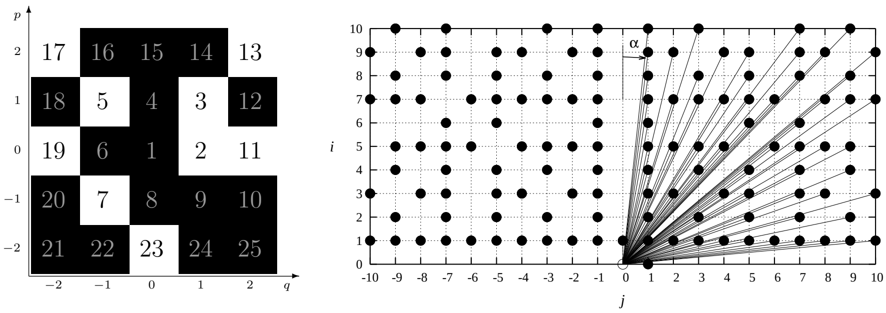
In this paper we shall shortly remind our previous findings, in order to pass to new results concerning the problem of looking for prime-rich polynomials, which we have found as a side-effect of looking for long sequences of points uniformly displaced in the Ulam square. With each sequence, a quadratic polynomial which generates its primes is connected. It occurred that one of the sequences found by us has a polynomial which is, in some sense, more rich in primes than some well-known polynomials which can be found in the literature.

At this point let us clarify the terminology. We shall call the *line* a set of points in the Ulam spiral forming a straight line, irrespectively of whether perceivable or not. A set of points located at fixed intervals and belonging to a line inclined at a specified angle will be called the *segment*. A segment has a limited number of points, naturally. We shall refer to this number as the *length* of the segment, in spite of that it is not the length in typical sense. The prime numbers corresponding to the points of the segment will be called the *sequence*. Conveniently, the *length* of this sequence is the same as the *length* of its segment. These terms will be further specified in the following.

The remainder of this paper is organized as follows. In Sect. 2 the method of detecting lines in the Ulam square will be outlined. In Sect. 3 the properties of the segments known until now will be described: their lengths and numbers in Sect. 3.1, directionality in Sect. 3.2, relation of number of segments to number of primes in Sect. 3.3, and some results for larger numbers and more directions in Sect. 3.4. These Sections contain a brief recapitulation of the results from our previous works on Ulam spiral, according to their sequence in time: [3], [4], [6], [5], [2]. In Section 4 we shall present our new findings concerning the prime-rich polynomials which we have come upon during the study. In the last Section the paper is summarized and concluded.

2 Detection of segments

Let us very briefly describe the basics of the applied line detection method proposed in [3]. Its description was partly repeated in [2,4,5,6]. We shall summarize and abbreviate the explanation from [2].



a **b**

Fig. 1. (**a**) The central part of the Ulam spiral for dimensions 5×5 with coordinates (p, q). Primes: black on white, other numbers: grey on black. (**b**) Directional vectors represented by the *direction table* D with elements D_{ij} containing increments $(\Delta p, \Delta q) = (i, j)$. Each vector has the initial point at the empty circle $(0, 0)$ and the terminal point at one of the black circles (i, j), $\neg(i = 0 \wedge j = 0)$. Angle α is the angle between the line segment and the vertical axis. From [2] and [3], with permissions.

Let us remind first the structure of the Ulam spiral shown in Fig. 1a. It starts with 1 as the center and goes on right, up, left, down and so on. Prime numbers are marked (here, in white). Other numbers remain on black. Some line segments present in the square are shown in Fig. 2a. To specify such objects let us establish a coordinate system Opq centered in the number 1. Let us consider three points corresponding to numbers $23, 7, 19$ which form a contiguous segment. Its slope, or direction, can be described by the differences in coordinates between its ends: $\Delta p = p(19) - p(32) = 2$ and $\Delta q = q(19) - q(32) = 2$. The directions are be stored in the table shown in Fig 1b called the *direction table*, denoted D, with elements D_{ij}, where $i = \Delta p$ and $j = \Delta q$. The considered segment can be described by the directional vector $(i, j) = (-2, 2)$, which can be reduced to $(-1, 1)$ and stored in D. The offset of the segment can be represented by q of its section with axis Oq, in this case $q(19) = -2$. In table D it is possible to represent slopes expressed by pairs $i \in [0 : N]$ and $j \in [-N, N] \setminus \{0\}$. Now let us consider points $23, 2, 13$ which form a segment inclined by $(i, j) = (2, 1)$, with offset $q(2) = 1$. These points are not neighbors in the normal sense, but they are the closest possible at this direction; hence, the segment formed by them will be denoted as *contiguous*. The angle can take as many different values as is the number of black circles in Fig. 1b, shown with thin lines.

The direction table can be used as an accumulator in the Hough transform for straight lines passing exactly through the points in the Ulam spiral. The vote is a pair of points, so this is be a two-point HT. The neighborhood from which a second point is taken for each first point, is related to the dimensions of D (doubling the pairs should be avoided). During the accumulation process, in each D_{ij} a one-dimensional data structure is formed. For each vote the line offset and the locations of two voting points are stored and the pairs and their primes are counted. After the accumulation, the accumulator can be analyzed according to

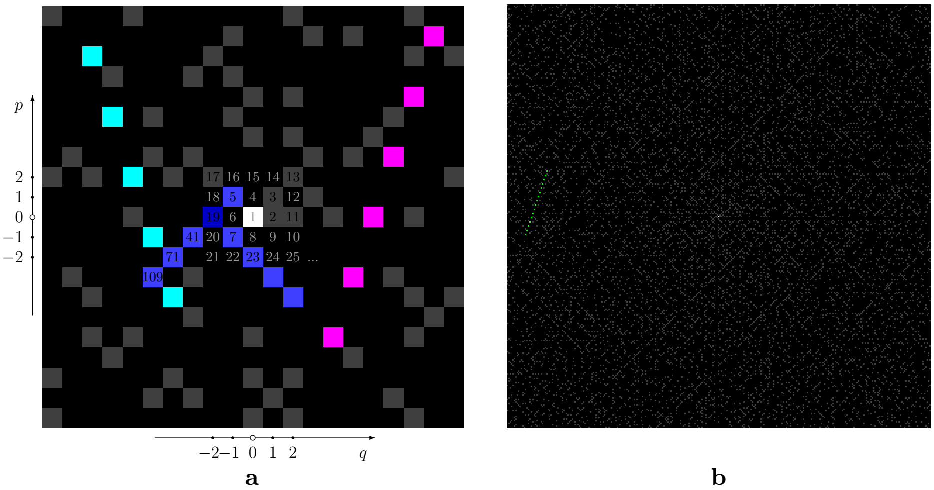


Fig. 2. (**a**) Central part of the Ulam spiral of dimensions 21×21 and with segments of lengths 5 and 6 marked in colors. Coordinates as in Fig. 1a. Primes: black on grey or color background; other numbers: grey on black or white. Blue: two segments of length 5, slopes $(\Delta p, \Delta q) = (1, 1)$ and $(1, -1)$, respectively, with one common point corresponding to prime number 19 marked with darker blue. Cyan: segment of length 5, slope $(\Delta p, \Delta q) = (3, -1)$. Magenta: segment of length 6, slope $(3, 1)$. From [5], with permission. (**b**) The longest, 16-point segment in 301×301 square, in green. From [4], with permission.

the need, for example, to find long segments or to find the dominating directions of lines. As an an example, in Fig. 2b the long segment having 16 primes and slope $(i, j) = (3, 1)$ is shown.

3 Properties of the detected segments

3.1 Number and length

It can be expected that short segments are more numerous in the Ulam square than long ones. Questions arise, how many segments of various lengths are there? Is there a limit on segment length? A partial answer to these and other questions can be seen in Fig. 3.

In general, the number of segments falls down with their length, not very far from linearly in log scale. It is interesting to see that there are no longer segments than 16 points, and that in the investigated set there are no segments of length 14 and 15 points [4] (this gap will be briefly addressed in Sect. 3.4).

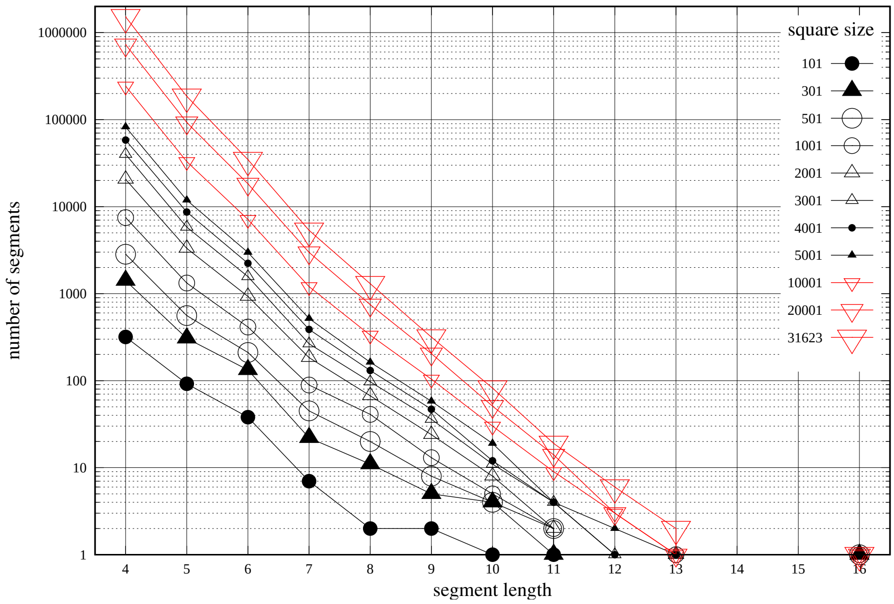


Fig. 3. Number of segments versus length of the segment for various sizes of the Ulam square, for direction array $N = 10$. If the number of lines is zero the data point is absent. The black lines show the results up to the largest prime 25 009 991 [4], the red ones – those for primes up to 1 000 014 121 [2]. From [2], with permission.

3.2 Directionality

It is known that there are more lines inclined at $(i, j) = (1, -1)$, angle $\alpha = -45°$ than at $(i, j) = (1, 1)$, $\alpha = 45°$ (angle α has been marked in Fig. 1b). This observation has been checked in [6].

The stronger the line is, the more voting pairs has voted for it. In Fig. 4 the numbers of votes on all the lines in the given direction is graphed. The numbers are normalized so that the maximum is one. It can be seen that the value for $\alpha = -45°$ is the largest, and that for $\alpha = 45°$ is the second large. The values for vertical ($\alpha = 0°$) and horizontal lines ($\alpha = 90°$) are smaller. It has been also checked if the directional structure of the Ulam square is indeed so specific, by comparing it with that of a random dot square, with the number of points corresponding to that of the Ulam square of the same size (cf. Fig. 6). It can be seen in the graphs in Fig. 4 that the random square has a different directional structure. One of the clearly visible differences is that the direction $\alpha = -45°$ is not privileged.

It has been also shown in [6] that the graphs in Fig. 4 stabilize quickly with the growing size of the square. The results for random data are practically indiscernible for their various realizations. Here, the results for squares $1001{*}1001$ are shown.

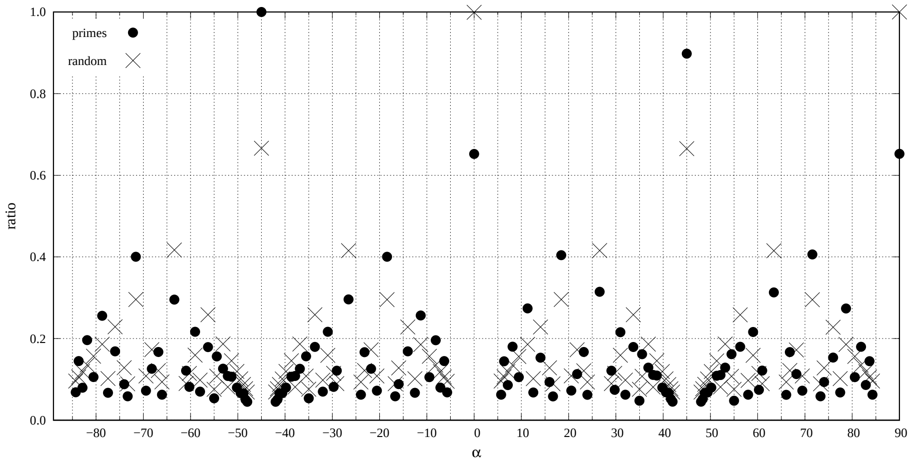


Fig. 4. Normalized numbers of votes (pairs of primes) on all the lines in the given direction, for Ulam square with primes and for random square with point density as in the Ulam one. From [6], modified, with permission.

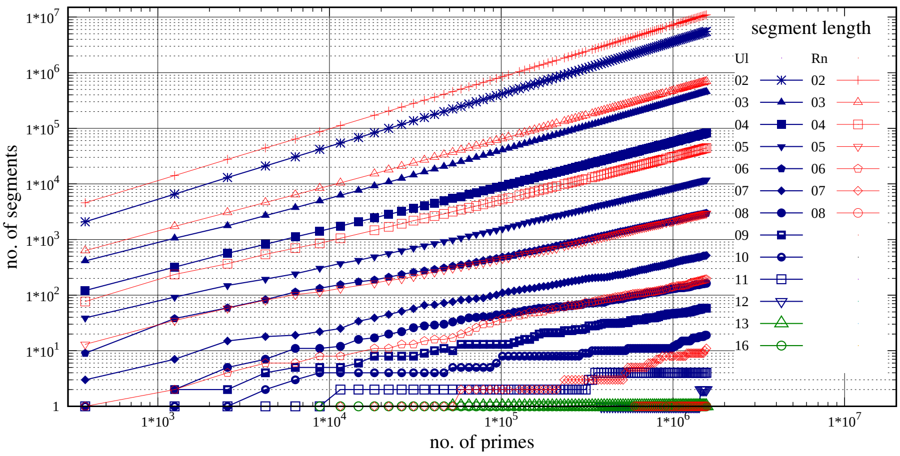


Fig. 5. Number of segments vs. number of primes in the Ulam square (Ul: thick blue or green lines, full symbols for lengths present also in random squares) and in random square (Rn: thin red lines, empty symbols). From [5], with permission.

3.3 Number of segments, number of primes

The primes are decreasingly dense in the set of natural numbers, as expressed by the function $\pi(n)$ [1,18]. It can be expected that the lesser the primes, the lesser the segments. The question arises whether the number of segments fall down quicker or slower than the number of primes decreases [5]. This relation is shown in Fig. 5 for all the lengths of segments present in the images.

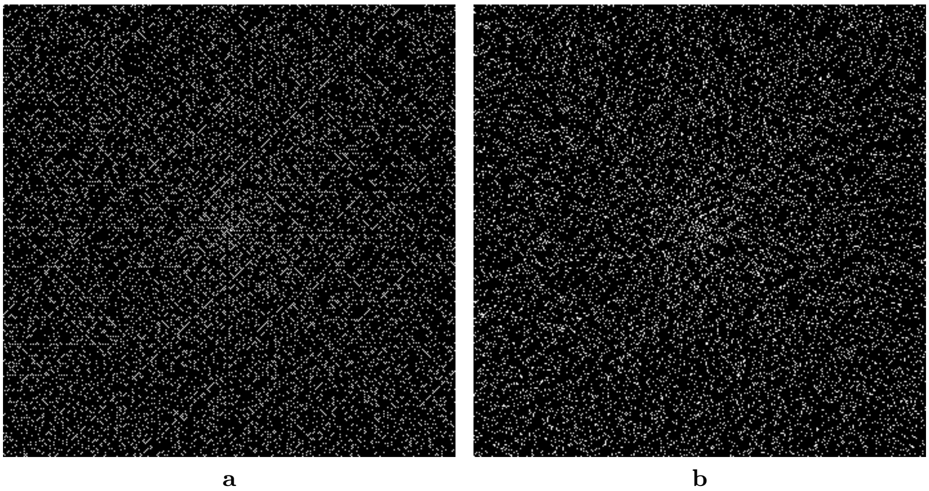


Fig. 6. (**a**) The actual Ulam square of size 351×351 and (**b**) one of its randomized realizations with density of white points corresponding to that in the Ulam square. The structure of the randomized square differs from that of the actual one even in visual inspection. From [5], with permission.

It can be seen that this relation is always close to linear in the double log scale, provided that the number of segments involved is large. The longest segments are rare and for them the relation does not hold.

Regular structures necessitate not only for the presence of the primes, but also for the existence of some constant relations between them. This is a restrictive demand, so it might well be expected that at some point the number of regular structures could fall down. No such phenomenon is observed in the range of primes investigated until now.

The human eye has a tendency of seeing regularities even in a random image. It is interesting whether the regularities just presented could not emerge from mere abundance of data. Random images having the density of dots very close to that observed in the Ulam square [5] (Fig. 6) can be subjected to the same analysis as the Ulam square. The results for randomized data can be seen in the Fig. 5. The sets of graphs appear different. Let us point out just two differences.

The longest segment in the Ulam square has 16 points, while in the considered realizations of the random square it has 8 points. This suggests that in the Ulam square the regularity is stronger than that in a random square.

In the random square the numbers of shorter segments (with 2 and 3 points) are larger than those in the Ulam square, while the numbers of the longer segments (with 4, 5, 6 points) is smaller, independently from the size of the squares. This suggests the existence of some factor which promotes the appearance of longer structures in the Ulam square.

It has been checked [5] that the results are stable for shorter segments, although the locations of segments in the subsequent realizations are entirely different.

The described observations suggest that the structure of the set of the prime numbers might contain long-range relations which do not depend on the scale.

3.4 Larger numbers, more directions

All the above described experiments (except those shown in red in Fig. 3) were carried out for the square of 5001×5001, hence for numbers up to approx. $25{\times}10^6$, and for the directions limited by the number $N = \max(i, j) = 10$. This was due to memory limits in the software and hardware. With the new equipment, the previously found tendencies were confirmed in [2] where numbers up to approximately 10^9 were analyzed. After extending the graphs from Fig. 5 to larger numbers the same tendencies were confirmed (see [2] for details). New interesting results have been found when the limit for the number of directions, previously set with the dimensions of the direction table $N = 10$, were relaxed by setting to $N = 50$ (with the size of the square 5001×5001, as it was in the first experiments). The results are shown in Fig. 7.

Several observations can be made. There are three new segments having 14 points. Two of them emerged near 3×10^5, and the third one for the primes over 10^6. Consequently, the gap between the segments with 13 and 16 points has been partly filled. There are also more segments with 13 points. Their points are much farther from each other than in the segments previously found.

Further, the close-to-linear shape of the graphs became less apparent in parts of the graphs. The linearity seems to hold for numbers over 10^5, quite similarly as

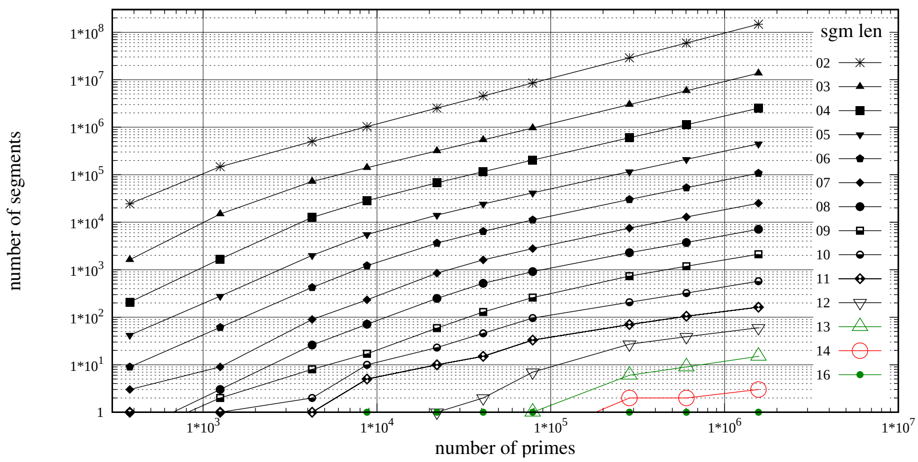


Fig. 7. Number of segments versus number of primes in the square, for direction array $N = 50$. From [2], with permission.

it was in the case of less directions. Extending the calculations to larger numbers could reveal the validity of the experimental asymptotic tendency in this case.

4 Prime-rich polynomials

It is quite easy to verify that many sequences of prime numbers of a given length (say l) discovered in the Ulam square, i.e., forming a segment of a detected line, can be generated by a special class of quadratic polynomials. In other words, there is a function of the form

$$f(n) = 4n^2 + bn + c \ ,$$

where b and c are some integer numbers, producing prime numbers belonging to the considered sequence for consecutive values of non-negative integers n from the range $0 \leq n \leq l-1$. For example, one of our detected sequences of eleven primes: 97571, 96323, 95083, 93851, 92627, 91411, 90203, 89003, 87811, 86627, 85451, which belongs to a line with the directional vector $(i, j) = (3, 1)$, is generated for non-negative integer numbers $0 \leq n \leq 10$ by the following quadratic polynomial:

$$a_{11}(n) = 4n^2 - 1260n + 98827 \ .$$

Finding polynomials (or any function, for that matter) generating prime numbers for subsequent values of arguments (being natural or integer numbers) is a very important problem for its own sake and its origins have nothing to do with the Ulam square, see, e.g., [9,14,15,16,20] and references therein. There is a famous example, known as the Euler quadratic polynomial

$$e(n) = n^2 + n + 41$$

that produces 40 different primes for non-negative integer numbers $0 \leq n \leq 39$. It is somewhat interesting that exactly this polynomial was invented by Lagrange, not Euler, original Euler's invention being a similar function: $g(n) = n^2 - n + 41$. In spite of the slightly different form both generate the same 40 distinct primes (in the latter case we should take $1 \leq n \leq 40$). As far as we know there is no other quadratic polynomial that improves over the Euler one in producing more than 40 different primes for subsequent values of non-negative integer arguments. However, it can happen that there are other quadratic polynomials that generate more primes in any given range of arguments than the Euler polynomial, if we relax the requirement that all these primes must come for subsequent values of arguments within that range. In fact, searching for such rich in primes quadratic polynomials has a long and exciting history.

Let us denote by $P_f(N)$ the number of primes generated by a given polynomial $f(n)$ for non-negative integer arguments $0 \leq n \leq N$. It is interesting to observe that for $N = 1000$ our polynomial $a_{11}(n)$ gives $P_{a_{11}}(1000) = 613$ primes (20 of them come with the minus sign) what is not only better than provided by the Euler polynomial $e(n)$, which gives $P_e(1000) = 582$, but better than several other well-known prime-rich polynomials like, e.g., all three so-called Beeger polynomials: $b_1(n) = n^2 + n + 19421$ (with $P_{b_1}(1000) = 558$), $b_2(n) = n^2 + n + 27941$ (with $P_{b_2}(1000) = 600$), and $b_3(n) = n^2 + n + 72491$ (with $P_{b_3}(1000) = 611$). Asymptotically, for larger N, our polynomial loses in this respect to the two latter polynomials, still beating both the Euler one and the first Beeger one [8,10,13]. Preliminary computations suggest that our polynomial has higher formally defined prime density than the Euler one and many other prime-rich polynomials. This is remarkable, especially taking into account the fact that polynomials discovered just by looking at segments of primes in the Ulam square were in no way optimized for that purpose, which follows from the descriptions given in the previous Sections. Of course not all quadratic polynomials generating sequences of primes forming line segments have these properties. Nevertheless the fact that starting from image analysis and pattern recognition perspectives we can obtain some potentially interesting results related to the number theory is, in our opinion, worth to be emphasized.

5 Conclusions

The set of prime numbers was considered from the perspective of its visualization in the form of the Ulam spiral. The objects of particular interest were the line segments, some of them perceivable with the human eye, and all detectable with image processing techniques. The detection method used was a specially designed version of the Hough transform. The search included the segments inclined at angles defined by ratios of integers up to 50, and the integers were up to approximately 10^9.

The results described in the previous papers were briefly recapitulated, including the numbers and lengths of the segments, their directionality, and the relation of their numbers to the number of primes.

It was shown that among the polynomials which generate the numbers forming the sequences found, at least one is exceptionally rich in primes. This polynomial generates one of the 11-point segments. This segment is not the longest segment found (the longest segment has 16 primes), and from the point of view of its appearance in the Ulam square it is not specific in any way. Its polynomial is $a_{11}(n) = 4n^2 - 1260n + 98827$ and among the numbers it generates for integer n, $0 \le n \le 1000$, there are 613 primes, including 20 of them with the minus sign. This is more than the number of primes generated by some other well-known prime-rich polynomials, like the Euler one and the three Beeger ones, for example. Asymptotically, this polynomial is more prime-rich than the Euler one and the first Beeger one. This is remarkable, because it has been discovered by looking for long segments in the Ulam square and not by optimizing its richness in primes.

This indicates that approaching the problem related to the set of prime numbers from the image analysis and pattern recognition perspective can lead to interesting results related to the number theory.

References

1. Caldwell, C.K.: How many primes are there? In: Caldwell, C.K. (ed.) The Prime Pages (2018), http://primes.utm.edu/howmany.html [Accessed 15 Jul 2018]
2. Chmielewski, L.J., Janowicz, M., Gawdzik, G., Orłowski, A.: Contiguous line segments in the Ulam spiral: Experiments with larger numbers. In: Rutkowski, L., Korytkowski, M., Scherer, R., Tadeusiewicz, R., Zadeh, L., Zurada, J. (eds.) Artificial Intelligence and Soft Computing: Proc. ICAISC 2017. Lecture Notes in Artificial Intelligence, vol. 10245, pp. 463–472. Springer, Cham, Zakopane, Poland (11-15 Jun 2017), doi:10.1007/978-3-319-59063-9_41
3. Chmielewski, L.J., Orłowski, A.: Hough transform for lines with slope defined by a pair of co-primes. Machine Graphics & Vision **22**(1/4), 17–25 (2013), [Open access: http://mgv.wzim.sggw.pl/MGV22.html#1-3]
4. Chmielewski, L.J., Orłowski, A.: Finding line segments in the Ulam square with the Hough transform. In: Chmielewski, L.J., Datta, A., Kozera, R., Wojciechowski, K. (eds.) Computer Vision and Graphics: Proc. ICCVG 2016. Lecture Notes in Computer Science, vol. 9972, pp. 617–626. Springer, Warsaw, Poland (19-21 Sep 2016), doi:10.1007/978-3-319-46418-3_55
5. Chmielewski, L.J., Orłowski, A., Gawdzik, G.: Segment counting versus prime counting in the Ulam square. In: Nguyen, N., Tojo, S., et al. (eds.) Proc. 9th Asian Conference on Intelligent Information and Database Systems ACIIDS 2017, Part II. Lecture Notes in Artificial Intelligence, vol. 10192, pp. 227–236. Springer, Kanazawa, Japan (3-5 Apr 2017), doi:10.1007/978-3-319-54430-4_22
6. Chmielewski, L.J., Orłowski, A., Janowicz, M.: A study on directionality in the Ulam square with the use of the Hough transform. In: Kobayashi, S., Piegat, A., et al. (eds.) Hard and Soft Computing for Artificial Intelligence, Multimedia and Security: Proc. Conf. ACS 2016. Advances in Intelligent Systems and Computing, vol. 534, pp. 81–90. Springer, Cham, Międzyzdroje, Poland (19-21 Oct 2016), doi:10.1007/978-3-319-48429-7_8
7. Cyganski, D., Noel, W.F., Orr, J.A.: Analytic Hough transform. In: Proc. SPIE. vol. 1260: Sensing and Reconstruction of Three-Dimensional Objects and Scenes, pp. 148–159 (1990), doi:10.1117/12.20013
8. Fung, G.W., Williams, H.C.: Quadratic polynomials which have a high density of prime values. Mathematics of Computation **55**(191), 345–353 (1990), doi:10.1090/S0025-5718-1990-1023759-3
9. Guy, R.K.: Unsolved Problems in Number Theory. Springer Science+Business Media, New York (1994)
10. Jacobson, M.J., Williams, H.C.: New quadratic polynomials with high densities of prime values. Mathematics of Computation **72**(241), 499–519 (2002), doi:10.1090/S0025-5718-02-01418-7
11. Kiryati, N., Lindenbaum, M., Bruckstein, A.M.: Digital or analog Hough transform? Pattern Recognition Letters **12**(5), 291–297 (1991), doi:10.1016/0167-8655(91)90412-F
12. Liu, Y., Cyganski, D., Vaz, R.F.: Efficient implementation of the analytic Hough transform for exact linear feature extraction. In: Proc. SPIE. vol. 1607: Intelligent Robots and Computer Vision X: Algorithms and Techniques, pp. 298–309 (1992), doi:10.1117/12.57109
13. Mollin, R.A.: Prime-producing quadratics. The American Mathematical Monthly **104**(6), 529–544 (1997), doi:10.1080/00029890.1997.11990675

14. Narkiewicz, W.: The Development of Prime Number Theory. From Euclid to Hardy and Littlewood. Springer-Verlag, Berlin, Heidelberg (2000)
15. Ribenboim, P.: The Book of Prime Number Records. Springer-Verlag, New York, second edn. (1989), doi:10.1007/978-1-4684-0507-1
16. Riesel, H.: Prime Numbers and Computer Methods for Factorization. Springer Science+Business Media, New York, second edn. (1994), originally published by Birkhäuser, Bosto,n as Vol. 126 in the series *Progress in Mathematics*. doi:10.1007/978-1-4612-0251-6
17. Stein, M.L., Ulam, S.M., Wells, M.B.: A visual display of some properties of the distribution of primes. The American Mathematical Monthly **71**(5), 516–520 (1964), doi:10.2307/2312588
18. Weisstein, E.W.: Prime counting function. From MathWorld–A Wolfram Web Resource (2018), http://mathworld.wolfram.com/PrimeCountingFunction.html [Accessed 15 Jul 2018]
19. Weisstein, E.W.: Prime spiral. From MathWorld–A Wolfram Web Resource (2018), http://mathworld.wolfram.com/PrimeSpiral.html [Accessed 15 Jul 2018]
20. Wells, D.: Prime Numbers. The Most Mysterious Figures in Math. John Wiley & Sons, Inc., Hoboken, New Jersey (2005)
21. Wikipedia: Ulam spiral — Wikipedia, The Free Encyclopedia (2016), https://en.wikipedia.org/wiki/Ulam_spiral [Accessed 15 Jul 2018]